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Exact Bianchi–Kantowski–Sachs solutions of Einstein's field equations

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Abstract. We present some old and many new exact solutions of the Einstein–Maxwell equations for the Bianchi type-III and Kantowski–Sachs space–times in a unique parametrisation. Solutions are given for the case of stiff matter and dust with (and without) a cosmological constant.

1. Introduction

The field equations of the general theory of relativity (GRT) for spatially homogeneous but anisotropic space–times have been investigated by many authors over the last thirty years since the basic work of Taub (1951). These space–times belong either to the Bianchi types I–IX or to the Kantowski–Sachs class and are often interpreted as cosmological models. There are two major distinct cases that arise: orthogonal universes, in which the matter moves orthogonally to the hypersurfaces of homogeneity, and tilted universes, in which the fluid vector $\boldsymbol{u}$ is not normal to such hypersurfaces. However, the Kantowski–Sachs models cannot be tilted. Considering only those solutions containing perfect fluid matter with equation of state $p = (\gamma - 1)\varepsilon$, $1 \leqslant \gamma \leqslant 2$ and (or) electromagnetic fields obeying the sourceless Maxwell equations or vacuum solutions in combination with a cosmological constant Λ, one would like to conjecture that almost all integrable cases have been found during the last three decades. (An almost complete list of solutions of Einstein's field equations is given by Kramer *et al* (1980).) However, in a systematic investigation of the combined Einstein–Maxwell equations for all spatially homogeneous models we have found that a number of cases have not been attacked previously.

Hughston and Jacobs (1970) (see also Jacobs 1977, Lorenz 1982b) have studied some properties of spatially homogeneous source-free electromagnetic fields in generic cosmological models with Bianchi symmetries. Their work was primarily stimulated by the apparent strength of a primordial intergalactic magnetic field of strength approximately 10^{-8} G (Kawabata *et al* 1969, Sofue *et al* 1969; see also Reinhardt and Thiel 1970, Reinhardt 1972, Kolobov *et al* 1976). However, no firm conclusion seems possible yet. The idea of a universe with a homogeneous magnetic field was proved to be very successful in flat (i.e. Bianchi type I) spaces (Zel'dovich and Novikov 1975, 1982). However, since Bianchi type I models are a very special subset of spatially homogeneous models, one should consider more general situations in order to check what implications large-scale primordial magnetic fields would have on the dynamics

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of the Universe. The basic work of Hughston and Jacobs has shown that the existence of a homogeneous primordial magnetic field in our Universe is limited to Bianchi types I, II, III, VI_0 and VII_0. The Kantowski–Sachs models are also allowed to contain a pure magnetic field (MacCullum 1979).

In papers published previously (Lorenz and Reinhardt 1980, Lorenz 1980a–d, 1981a–c, 1982a–k, Lorenz and Zimmermann 1981) we presented new exact solutions for various Bianchi and Kantowski–Sachs models. In this paper we consider magnetic Bianchi type-III models and the related Kantowski–Sachs space–time. Exact solutions are given for the case of stiff matter ($\gamma = 2$) and dust ($\gamma = 1$) with (and without) a cosmological constant Λ in a unique parametrisation. The cosmological implications of these solutions will be discussed in a future paper.

2. Field equations and solutions

The field equations to be solved are

$$3\dot{H} + H_1^2 + H_2^2 + H_3^2 = -\tfrac{1}{2}\varepsilon(3\gamma - 2)\cosh^2\beta - \tfrac{1}{2}\varepsilon(2-\gamma)\sinh^2\beta - (f/R_1R_2)^2 + \Lambda \tag{1a}$$

$$\dot{H}_1 + H_1^2 + H_1H_2 + H_1H_3 - \delta(2q/R_1)^2 = \tfrac{1}{2}\varepsilon(2-\gamma) + \varepsilon\gamma\sinh^2\beta + (f/R_1R_2)^2 + \Lambda \tag{1b}$$

$$\dot{H}_2 + H_2^2 + H_2H_1 + H_2H_3 - \delta(2q/R_1)^2 = \tfrac{1}{2}\varepsilon(2-\gamma) + (f/R_1R_2)^2 + \Lambda \tag{1c}$$

$$\dot{H}_3 + H_3^2 + H_3H_1 + H_3H_2 = \tfrac{1}{2}\varepsilon(2-\gamma) - (f/R_1R_2)^2 + \Lambda \tag{1d}$$

$$(H_1 - H_2)2q/R_1 = \varepsilon\gamma\cosh\beta\sinh\beta \tag{1e}$$

where $H_i = \dot{R}_i/R_i$ are the Hubble parameters ($H = \frac{1}{3}\Sigma_i H_i$), $R_i = R_i(t)$ are the cosmic scale functions, $f^2/(R_1R_2)^2$ are the components of the magnetic field ($f, q =$ constant) and the perfect fluid matter is characterised by the equation of state

$$p = (\gamma - 1)\varepsilon \qquad 1 \leqslant \gamma \leqslant 2 \tag{2}$$

where ε and p are, respectively, the density and pressure of the fluid. (A dot denotes differentiation with respect to t.) In addition to equations (1) we have the conservation equations

$$(\ln(\varepsilon^{1/\gamma}R_1R_2R_3\cosh\beta))^{\cdot} = (2/R_1)\tanh\beta \tag{3a}$$

$$(\ln(\varepsilon^{(\gamma-1)/\gamma}R_1\sinh\beta))^{\cdot} = 0 \tag{3b}$$

where β denotes the hyperbolic tilt angle. The case $\delta = 1$ corresponds to the Bianchi type-III model and for the Kantowski–Sachs model we have $\delta = -1/4q^2$ and $R_1 = R_2$.

We will now give a complete discussion of the field equations for the cases of stiff matter ($\gamma = 2$), dust ($\gamma = 1$), vacuum ($\varepsilon = 0$) and pure magnetic fields ($\varepsilon = 0, f \neq 0$). The solutions are classified according to the values of the parameters (ε, γ, f, β, Λ).

3. Bianchi type-III solutions

We first consider the case $\delta = 1$. Introducing the new time variable τ by $dt = R_1\,d\tau$ the linear combination of (1c) and (1d) gives

$$(R_2R_3)'' - 4q^2R_2R_3 = [\varepsilon(2-\gamma) + 2\Lambda]R_1^2R_2R_3 \tag{4a}$$

where $(\ )' = \mathrm{d}/\mathrm{d}\tau$, which can be easily integrated if $\Lambda = 0$ and $\gamma = 2$ or $\varepsilon = 0$. From (1*b*) and (1*d*) we obtain

$$[R_2R_3(\ln(R_1R_3))']' - 4q^2R_2R_3 = [\varepsilon(2-\gamma) + \varepsilon\gamma \sinh^2\beta + 2\Lambda]R_1^2R_2R_3. \tag{4b}$$

Once a first integral of (4*a*) has been obtained for R_2R_3, we can solve (4*b*) for R_1R_3. Finally, we have from (1*b*) and (1*c*)

$$[R_2R_3(\ln(R_1R_2))']' - 8q^2R_2R_3 = [\varepsilon(2-\gamma) + \varepsilon\gamma \sinh^2\beta + 2\Lambda]R_1^2R_2R_3 + f^2(R_3/R_2) \tag{4c}$$

to give R_1R_2. In addition, we have the constraint equation

$$(\ln R_1)'(\ln(R_2R_3))' + (\ln R_2)'(\ln R_3)' = [\varepsilon(\gamma\sinh^2\beta + 1) + \Lambda]R_1^2 + 4q^2 + (f/R_2)^2. \tag{4d}$$

We now quote all known and some new solutions. The tilted stiff matter solution $(\varepsilon, 2, 0, \beta, 0)$ was found by Wainwright *et al* (1979):

$$R_1^2 = (\sinh 2q\tau)^{2(2b^2+1)}(\tanh q\tau)^{km+4ab} \tag{5a}$$

$$R_2^2 = (\sinh 2q\tau)^2(\tanh q\tau)^{km} \tag{5b}$$

$$R_3^2 = (\tanh q\tau)^{-km} \tag{5c}$$

$$m^2 - 4 + 4(a^2 - b^2) = 0 \tag{5d}$$

where a and b are constants of integration and $k^2 = 1$. The energy density of matter is given by

$$\varepsilon = \frac{4q^2}{R_1^2 \sinh^2 2q\tau}(a^2 + 2ab\cosh 2q\tau + b^2) \tag{5e}$$

and the hyperbolic tilt angle β can be found from (1*e*) to be

$$\coth\beta = (1/\sinh 2q\tau)[\cosh 2q\tau + (a/b)]. \tag{5f}$$

The non-tilted solution $(\varepsilon, 2, 0, 0, 0)$ is given by $b = 0$ and was found by Ruban (1978) and, in other forms, by Kantowski (1966), Vajk and Eltgroth (1970), Collins (1971) and Tsoubelis (1981). In the case of $a = b = 0$ the solution is reduced to the vacuum case (0, 0, 0, 0, 0) first given by Ellis and MacCallum (1969). The tilted magnetic solution $(\varepsilon, 2, f, \beta, 0)$ has been found by us recently (Lorenz 1982c) and can be presented in the form

$$R_1 = \frac{f}{2q}(\cosh 2q\tau)(\sinh 2q\tau)^{2b^2}(\tanh q\tau)^{2kb^2} \tag{6a}$$

$$R_2 = \frac{f}{2q}\cosh 2q\tau \tag{6b}$$

$$R_3 = \frac{2q}{f}\tanh 2q\tau \tag{6c}$$

where $kb = a$. The equations (5*e*) and (5*f*) for ε and β remain the same. (Note that equations (1*e*) and (11) of Lorenz (1982*c*) contain an error of sign.) We would like to point out that our solution does not reduce to the above given stiff matter solution when $f = 0$. By setting $b = 0$ we obtain the pure magnetic solution $(0, 0, f, 0, 0)$ which has been previously found by Vajk and Eltgroth (1970) in a different parametrisation.

The non-tilted magnetic stiff matter solution $(\varepsilon, 2, f, 0, 0)$ is given by

$$R_1 = R_2 = \frac{f}{2m}\left(\frac{1+(\tanh q\tau)^{m/2q}}{(\tanh q\tau)^{m/2q}}\right)(\sinh 2q\tau) \tag{7a}$$

$$R_3 = 2\frac{m}{f}\frac{(\tanh q\tau)^{m/2q}}{1+(\tanh q\tau)^{m/q}} \tag{7b}$$

$$m^2 - 4q^2 + \varepsilon_0^2 = 0 \tag{7c}$$

with $\varepsilon = \varepsilon_0^2/(R_1^2 R_3)^2$. This solution was also found by Tsoubelis (1981) in another form and reduces to the pure magnetic case if $\varepsilon_0^2 = 0$. Finally we would like to point out that it seems to be impossible to integrate the field equations (1) with a non-vanishing cosmological constant Λ in the stiff matter case.

We now turn to the case of dust $(\gamma = 1)$. The tilted magnetic dust solutions $(\varepsilon, 1, f, \beta, \Lambda)$ and $(\varepsilon, 1, f, \beta, 0)$ can be reduced to (9) (see below) and two second-order differential equations for R_2 and R_2R_3. However, to construct the explicit solutions remains a problem for the near future. The tilted dust solution $(\varepsilon, 1, 0, \beta, 0)$ can be obtained in the following way. From the conservation equations $(3a)$ and $(3b)$ we obtain

$$R_1 = \frac{a}{\sinh \beta} \qquad \varepsilon = \frac{b}{R_2R_3}\cosh\beta \sinh\beta \qquad a, b = \text{constant}. \tag{8}$$

Substitution of (8) into the linear combination of $(1b)$, $(1c)$ and $(1e)$ yields a first-order ordinary differential equation for R_1 whose solution is

$$R_1 = a/\sinh q\tau. \tag{9}$$

From $(4a)$ we obtain with the method of variation of parameters the solution

$$R_2R_3 = \frac{a^2 b}{2q^2}[(\tfrac{1}{4} + \ln \sinh q\tau)\sinh 2q\tau - q\tau \cosh 2q\tau] + A\sinh 2q\tau + B\cosh 2q\tau \tag{10}$$

where A and B are constants of integration. Equations $(1a)$–$(1d)$ are rearranged to give

$$R_3''/R_3 = -(R_1'/R_1)' = -q^2/\sinh^2 q\tau. \tag{11}$$

Applying the transformations

$$R_3(\tau) = \eta(z) \qquad z = \cosh^2 q\tau \qquad (\ )^{\cdot} = \mathrm{d}/\mathrm{d}z \tag{12}$$

we obtain the differential equation

$$z(1-z)\ddot{\eta} + (\tfrac{1}{2} - z)\dot{\eta} - \frac{\eta}{4(1-z)} = 0. \tag{13}$$

Putting

$$\eta = (1-z)^{\nu/2} f(z) \qquad \nu(\nu - 1) = 1 \tag{14}$$

we arrive at the hypergeometric equation

$$z(1-z)\ddot{f} + [\tfrac{1}{2} - (\nu+1)z]\dot{f} - \tfrac{1}{4}\nu^2 f = 0 \tag{15}$$

which can be solved in terms of hypergeometric functions. For R_2 we could solve

$$R_2''/R_2 = -(R_1'/R_1)' + 4q^2 \tag{16}$$

directly by a very similar calculation instead of using (10). If $\Lambda \neq 0$ equation (11) is to be replaced by

$$R_3''/R_3 = (\Lambda a^2 - q^2)/\sinh^2 q\tau. \tag{17}$$

Introducing the new variables

$$R_3(\tau) = \eta(\xi) \qquad q\tau = \pm\ln[\xi/(1+\xi)]^{1/2} \qquad \xi > 0 \tag{18}$$

we obtain

$$\ddot{\eta}\xi(1+\xi) + \dot{\eta}(2\xi+1) - 4\frac{(\Lambda a^2 - q^2)}{q^2}\eta = 0 \tag{19}$$

which becomes a hypergeometric equation by

$$y(x) = \eta(\xi) \qquad \xi = -x.$$

After solving

$$\frac{R_2''}{R_2} = \frac{\Lambda a^2 - q^2}{\sinh^2 q\tau} + 4q^2 \tag{20}$$

in a similar manner for $R_2 = R_2(\tau)$ the case $(\varepsilon, 1, 0, \beta, \Lambda)$ is determined by (9), (18), (19) and (20).

The non-tilted dust solutions can be obtained easily from

$$2\frac{R_1''}{R_1} - \left(\frac{R_1'}{R_1}\right)^2 - 4q^2 + [\varepsilon(\gamma-1) - \Lambda]R_1^2 - \left(\frac{f}{R_2}\right)^2 = 0 \tag{21a}$$

$$\frac{R_3'}{R_3}\frac{R_1'}{R_1} = \left(\frac{R_1'}{R_1}\right)' + \tfrac{1}{2}\varepsilon\gamma R_1^2 \tag{21b}$$

$$R_1'/R_1 = R_2'/R_2. \tag{21c}$$

In the case $R_1' = 0$ we obtain the special vacuum solution $(0, 0, f, 0, \Lambda)$ with $R_3 = R_3(\tau)$ arbitrary. The dust solution $(\varepsilon, 1, 0, 0, \Lambda)$ has been given by Ellis (1967) (see also Kramer *et al* 1980). If $R_1' \neq 0$ equation (21*a*) is a Bernoulli-type differential equation, the solution of which can be written for $\varepsilon = 0$ or $\gamma = 1$ ($y = R_1/2q$):

$$2q\tau = \int [\tfrac{1}{3}\Lambda y^4 + y^2 + cy - (f/4q^2)^2]^{-1/2}\, dy. \tag{22}$$

Equation (22) was first given by Stewart and Ellis (1968) for the case $\varepsilon = 0$. The integration can be performed by elementary functions if $c = 0$, $\Lambda = 0$ or $f = 0$. The general dust solution $(\varepsilon, 1, 0, 0, \Lambda)$ with $c \neq 0$ has been found by us very recently (Lorenz 1982h). If $R_1 = c\xi$, $q = \frac{1}{2}$ then ξ satisfies

$$(d\xi/d\tau)^2 = \Omega\xi^4 - k\xi^2 + \xi \tag{23}$$

where $\Omega = \frac{1}{3}\Lambda c^2$ and $k = -1$. The solution of this equation with $\xi = 0$ at $\tau = 0$ can be written as

$$\xi(\tau) = 3(3\wp(\tfrac{1}{2}\tau; g_2, g_3) + k)^{-1} \tag{24}$$

where $\wp$ is the Weierstrass elliptic function which involves the two parameters g_2 and g_3 and satisfies

$$(d\wp/d\tau)^2 = 4\wp^3 - g_2\wp - g_3. \tag{25}$$

Substituting in the differential equation (23) we find that (24) is a solution if

$$g_2=\tfrac{4}{3} \qquad g_3=4(\tfrac{2}{27}k-\Omega). \tag{26}$$

The function ϕ can be expressed in terms of elliptic functions, the exact relationship depending on the sign of the discriminant:

$$\Delta:=g_2^3-27g_3^2=\tfrac{16}{27}(2-2k+27\Omega)(2+2k-27\Omega). \tag{27}$$

The corresponding solution for R_3 can be obtained from (21b). We find
(i) $\Delta<0$:

$$R_1=c\frac{1-\mathrm{cn}(\alpha\tau,k_1)}{\alpha^2+e+(\alpha^2-e)\,\mathrm{cn}(\alpha\tau,k_1)} \tag{28a}$$

$$\begin{aligned}R_3=&\frac{2\alpha^3c\,\mathrm{sn}(\alpha\tau,k_1)\,\mathrm{dn}(\alpha\tau,k_1)}{(1-\mathrm{cn}(\alpha\tau,k_1))[\alpha^2+e+(\alpha^2-e)\,\mathrm{cn}(\alpha\tau,k_1)]}\\&\times\left\{a+\frac{\varepsilon_0^2}{8\alpha^7}\left[\frac{2ek(\alpha^2-e)}{\alpha k_1^2}\tau-8e^2\left(E(2\varphi,k_1)+\frac{\mathrm{sn}(2\alpha\tau,k_1)\,\mathrm{dn}(2\alpha\tau,k_1)}{1-\mathrm{cn}(2\alpha\tau,k_1)}\right)\right.\right.\\&+\left(\frac{(\alpha^2+e)^2}{1-k_1^2}+\frac{(\alpha^2-e)^2}{k_1^2}\right)E(\varphi,k_1)+4e(\alpha^2+e)\frac{\mathrm{dn}(\alpha\tau,k_1)}{\mathrm{sn}(\alpha\tau,k_1)}\\&\left.\left.-\left(\frac{(\alpha^2+e)^2}{1-k_1^2}k_1^2-(\alpha^2-e)^2\right)\frac{\mathrm{sn}(\alpha\tau,k_1)\,\mathrm{cn}(\alpha\tau,k_1)}{\mathrm{dn}(\alpha\tau,k_1)}-8e^2k_1^2\frac{\mathrm{sn}(\alpha\tau,k_1)}{\mathrm{dn}(\alpha\tau,k_1)}+b\right]\right\};\end{aligned} \tag{28b}$$

(ii) $\Delta>0$:

$$R_1=c\frac{1-\mathrm{cn}(\beta\tau,k_2)}{e(1-\mathrm{cn}(\beta\tau,k_2))+\beta^2(1+\mathrm{dn}(\beta\tau,k_2))} \tag{28c}$$

$$\begin{aligned}R_3=&\frac{\beta^3\,\mathrm{sn}(\beta\tau,k_2)[1+\mathrm{dn}(\beta\tau,k_2)-k_2^2(1-\mathrm{cn}(\beta\tau,k_2))]}{(1-\mathrm{cn}(\beta\tau,k_2))[e(1-\mathrm{cn}(\beta\tau,k_2))+\beta^2(1+\mathrm{dn}(\beta\tau,k_2))]}\\&\times\left[a+\frac{\varepsilon_0^2}{2\beta^6}\int\left(\frac{e\,\mathrm{sn}^2(\frac{1}{2}\beta\tau,k_2)+\beta^2}{\mathrm{sn}(\beta\tau,k_2)\,\mathrm{dn}^2(\frac{1}{2}\beta\tau,k_2)}\right)^2\mathrm{d}\tau\right]\end{aligned} \tag{28d}$$

where

$$\alpha^4=e(3e-2k) \qquad 2\beta^2=k-3e+(1-3e^2+2ek)^{1/2} \tag{28e}$$

$$4k_1^2=2+(k-3e)\alpha^{-2} \qquad k_2^2=-1+(k-3e)\beta^{-2} \tag{28f}$$

$$\Omega=e^2(k-e) \tag{28g}$$

and $E(2\varphi,k_1)$, $E(\varphi,k_1)$ are elliptic integrals of the second kind with $\varphi=\mathrm{cn}^{-1}(\alpha\tau,k_1)$. If we regard Ω as being known then e is given by solving equation (28g); alternatively, for a given value of the parameter e, equations (28e) and (28f) determine all the relevant quantities and for each e we have a family of solutions. The energy density ε is given by $\varepsilon=\varepsilon_0^2/(R_1^2R_3)$, $\varepsilon_0^2=$ constant.

If $\varepsilon_0^2=0$ our solution reduces to the general vacuum solution $(0,0,0,0,\Lambda)$ first given by Cahen and Defrise (1968) in a different parametrisation. Equation (22) can be rewritten in the form

$$\tilde{f}(u)=(2mu-f^2+\tfrac{1}{3}\Lambda u^4+u^2)/u^2 \tag{29}$$

where $2m = c$, $q = \frac{1}{2}$, $u = R_1$. $\tilde{f}(u) = \dot{R}_1^2$ and $(\dot{\ }) = \mathrm{d}/\mathrm{d}t$. Equation (29) was first given by Ruban (1972, 1978). The special cases $f = 0$, $\Lambda \neq 0$ and $f \neq 0$, $\Lambda = 0$ have been also given by Cahen and Defrise (1968) (see also Kramer *et al* 1980). The special dust solutions $(\varepsilon, 1, 0, 0, \Lambda)$ with $c = 0$ are given by

(i) $\Lambda > 0$:

$$R_1 = R_2 = k'2q(3/\Lambda)^{1/2}(\sinh 2q\tau)^{-1} \qquad k'^2 = 1 \tag{30a}$$

$$R_3 = \left(a + \frac{\varepsilon_0^2}{8q^2}b\right)\coth 2q\tau + \frac{\varepsilon_0^2}{8q^2}(2q\tau \tanh 2q\tau - 1); \tag{30b}$$

(ii) $\Lambda < 0$:

$$R_1 = R_2 = 2q(3/-\Lambda)^{1/2}(\cosh 2q\tau)^{-1} \tag{30c}$$

$$R_3 = \left(a + \frac{\varepsilon_0^2}{8q^2}b\right)\tanh 2q\tau + \frac{\varepsilon_0^2}{8q^2}(2q\tau \tanh 2q\tau - 1) \tag{30d}$$

where a and b are constants of integration. If $\varepsilon_0^2 = 0$ our solutions reduce to the vacuum solutions $(0, 0, 0, 0, \Lambda)$ obtained recently by Moussiaux *et al* (1981) and are also contained in the general vacuum solution (29) as has been shown by MacCallum *et al* (1982).

The special magnetic dust solution $(\varepsilon, 1, f, 0, \Lambda)$ with $c = 0$ is given by

(i) $\Lambda > 0$:

$$R_1 = 2q[-\tfrac{3}{2}\Lambda + \tfrac{3}{2}\Lambda(1 + 12f^2\Lambda/q^4)^{1/2}]^{1/2}\,\mathrm{nc}(\alpha\tau, k_1) \tag{31a}$$

$$R_3 = \frac{\mathrm{sn}(\alpha\tau, k_1)\,\mathrm{dn}(\alpha\tau, k_1)}{\mathrm{cn}(\alpha\tau, k_1)}\left[a + \tfrac{1}{2}\varepsilon_0^2\left(\alpha\tau - E(2\alpha\tau, k_1) - \frac{\mathrm{sn}(2\alpha\tau, k_1)\,\mathrm{dn}(2\alpha\tau, k_1)}{1 - \mathrm{cn}(2\alpha\tau, k_1)} + b\right)\right]; \tag{31b}$$

(ii) $\Lambda < 0$:

$$R_1 = 2q[\tfrac{3}{2}\Lambda + \tfrac{3}{2}\Lambda(1 - 12f^2\Lambda/q^4)^{1/2}]^{1/2}\,\mathrm{nc}(\beta\tau, k_2) \tag{31c}$$

$$R_3 = \frac{\mathrm{sn}(\beta\tau, k_2)\,\mathrm{dn}(\beta\tau, k_2)}{\mathrm{cn}(\beta\tau, k_2)}\left[a + \tfrac{1}{2}\varepsilon_0^2\left(\beta\tau - E(2\beta\tau, k_2) - \frac{\mathrm{sn}(2\beta\tau, k_2)\,\mathrm{dn}(2\beta\tau, k_2)}{1 - \mathrm{cn}(2\beta\tau, k_2)} + b\right)\right] \tag{31d}$$

where

$$\alpha = 2q(1 + 12f^2\Lambda/q^4)^{1/4} \qquad \beta = 2q(-{}^{-}\Lambda)^{1/2}(-\Lambda^2 + 3f^2/16q^4\Lambda)^{1/4} \tag{31e}$$

$$k_1^2 = [2(\tfrac{9}{4}\Lambda^2 + 3f^2/q^4\Lambda)^{1/2} + 3/\Lambda]/4(\tfrac{9}{4}\Lambda^2 + 3f^2/q^4\Lambda)^{1/2} \tag{31f}$$

$$k_2^2 = [2(\tfrac{9}{4}\Lambda^2 - 3f^2/q^4\Lambda)^{1/2} - 3/\Lambda]/4(\tfrac{9}{4}\Lambda^2 - 3f^2/q^4\Lambda)^{1/2}. \tag{31g}$$

The general magnetic dust solution $(\varepsilon, 1, f, 0, 0)$ with $c = 1/q$ is given by

$$R_1 = R_2 = a\cosh 2q\tau - 1 \tag{32a}$$

$$R_3 = \frac{1}{R_1}\left(\sinh 2q\tau + \frac{\varepsilon_0^2}{8q^2a}[2qa\tau \sinh 2q\tau - (1 + a)\cosh 2q\tau + 2q]\right) \tag{32b}$$

where $a^2 = [1 + (f^2/4q^2)]$ and $\varepsilon = \varepsilon_0^2/R_1^2R_3$, $\varepsilon_0^2 = \text{constant}$. The solution given here is very similar to that given by Vajk and Eltgroth (1970). These expressions define the solutions first obtained by Doroshkevich (1965), Thorne (1966, 1967) and Shikin (1966) in yet another form. If ε_0^2 vanishes we obtain the $(0, 0, f, 0, 0)$ solution first given by Vajk and Eltgroth (1970) while for $\varepsilon_0^2 = f = 0$ the solution reduces to the vacuum case $(0, 0, 0, 0, 0)$(6a–c). The case $f = 0$ $(\varepsilon, 1, 0, 0, 0)$ has been also given by Kompaneets and Chernov (1964), Kantowski (1966) and Kantowski and Sachs (1966) in different forms.

Finally, we present the special magnetic dust solution $(\varepsilon, 1, f, 0, 0)$ with $c = 0$:

$$R_1 = 2q \exp(2q\tau) + (f^2/32q^3) \exp(-2q\tau) \tag{33a}$$

$$R_3 = 2q \frac{64q^4 \exp(2q\tau) - f^2 \exp(-2q\tau)}{64q^4 \exp(2q\tau) + f^2 \exp(-2q\tau)} \times \left[a + \frac{\varepsilon_0^2}{32q^3} \left(\ln\left(\frac{(64q^4 \exp(4q\tau) - f^2)^{1+f^2}}{(\exp(4q\tau))^{f^2}} \right) - \frac{64q^4 \exp(4q\tau) - 3f^2}{64q^4 \exp(4q\tau) - f^2} + b \right) \right]. \tag{33b}$$

If $f = 0$, it reduces to the special dust solution given by Ftaclas and Cohen (1979) in a different parametrisation.

4. Kantowski–Sachs solutions

Since there are no tilted Kantowski–Sachs models the solutions presented here are classified according to the parameters $(\varepsilon, \gamma, f, \Lambda)$. The results are as follows.

4.1. $(\varepsilon, 2, f, 0)$

$$R_1 = \frac{f}{2m} (\sin \tau) \frac{1 + (\tan \tau)^{2m}}{(\tan \tau)^m} \tag{34a}$$

$$R_3 = \frac{2m}{f} \frac{(\tan \tau)^m}{1 + (\tan \tau)^{2m}} \tag{34b}$$

$$m^2 - 1 + \varepsilon_0^2 = 0 \tag{34c}$$

where $\varepsilon = \varepsilon_0^2 / R_1^4 R_3^2$. This solution is new and includes the pure magnetic case $(0, 0, f, 0)$ first given by Vajk and Eltgroth (1970) in a somewhat different form.

4.2. $(\varepsilon, 2, 0, 0)$

$$R_1^2 = (\sin \tau)^2 (\tan \tfrac{1}{2}\tau)^{km} \tag{35a}$$

$$R_3^2 = (\tan \tfrac{1}{2}\tau)^{-km} \tag{35b}$$

$$m^2 - 4(1 - \varepsilon_0^2) = 0 \qquad k^2 = 1. \tag{35c}$$

Kantowski (1966) first obtained these solutions and they were rediscovered by Vajk and Eltgroth (1970) in a different form. For $\varepsilon_0^2 = 0$ the solution is reduced to the vacuum case $(0, 0, 0, 0)$ also given by Kantowski (1966). Until now no stiff matter solutions with $\Lambda \neq 0$ have been found. This is in contrast to the Bianchi type-I and type-V stiff matter solutions with $\Lambda \neq 0$ given by Ellis and MacCallum (1969).

4.3. $(\varepsilon, 1, f, \Lambda)$

The special magnetic dust solutions with $c=0$ are given by

$$(a)\quad R_1=[\tfrac{3}{2}\Lambda+\tfrac{3}{2}\Lambda(1+\tfrac{4}{3}f^2\Lambda)^{1/2}]^{1/2}\,\mathrm{ns}(\omega\tau,k) \tag{36a}$$

$$R_3=\frac{\mathrm{cn}(\omega\tau,k)\,\mathrm{dn}(\omega\tau,k)}{\mathrm{sn}(\omega\tau,k)}\bigg[a+\frac{\varepsilon_0^2}{2}\bigg(\frac{1}{k'^4}(k'^2\omega\tau-2E(\omega\tau,k)$$
$$+(1+k^2-2k^2\,\mathrm{sn}^2(\omega\tau,k))\,\mathrm{tn}(\omega\tau,k)\,\mathrm{nd}(\omega\tau,k))+b\bigg)\bigg] \tag{36b}$$

$$(b)\quad R_1=[\tfrac{3}{2}\Lambda+\tfrac{3}{2}\Lambda(1+\tfrac{4}{3}f^2\Lambda)^{1/2}]^{1/2}\,\mathrm{nc}(\omega\tau,k) \tag{36c}$$

$$R_3=\frac{\mathrm{sn}(\omega\tau,k)\,\mathrm{dn}(\omega\tau,k)}{\mathrm{cn}(\omega\tau,k)}\bigg[a+\frac{\varepsilon_0^2}{2}\bigg(\omega\tau-E(2\omega\tau,k)$$
$$-\frac{\mathrm{sn}(2\omega\tau,k)\,\mathrm{dn}(2\omega\tau,k)}{1-\mathrm{cn}(2\omega\tau,k)}+b\bigg)\bigg] \tag{36d}$$

where $\omega=(1+\frac{4}{3}f^2\Lambda)^{1/4}$, $k^2=[(1+\frac{4}{3}f^2\Lambda)^{1/2}+2]/(1+\frac{4}{3}f^2\Lambda)^{1/2}$ and $k'^2=1-k^2$. These solutions are new and reduce to the case $(\varepsilon, 1, 0, \Lambda)$, which corresponds to the Bianchi type-III solutions (30*a*–*d*). The Kantowski–Sachs dust solutions given here are derived from the corresponding integral (22) by transforming $y^2\to -y^2$. The energy density is given by $\varepsilon=\varepsilon_0^2/R_1^2R_3$. The most general solution $(\varepsilon, 1, 0, \Lambda)$ with $c\neq 0$ is given by (28*a*–*g*) with $k=1$ and can be also given in the form (29).

4.4. $(\varepsilon, 1, f, 0)$

We obtain two different kinds of solutions:

$$(a)\quad R_1=\tfrac{1}{2}(c+b\sin\tau) \tag{37a}$$

$$R_3=\frac{1}{c+b\sin\tau}\Big(d\cos\tau+\frac{\varepsilon_0^2}{2b}[(c^2+b^2)\sin\tau+2cb-b^2\tau\cos\tau]\Big) \tag{37b}$$

$$(b)\quad R_1=\tfrac{1}{2}(c+b\cos\tau) \tag{37c}$$

$$R_3=\frac{1}{c+b\cos\tau}\Big(d\sin\tau+\frac{\varepsilon_0^2}{2b}[(c^2+b^2)\cos\tau+2cb+b^2\tau\sin\tau]\Big) \tag{37d}$$

where $b=(c^2-4f^2)^{1/2}$. The first one is entirely new. The second has been first obtained by Doroshkevich (1965), Thorne (1966, 1967), Shikin (1966) and Vajk and Eltgroth (1970) in different parametrisations.

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